Stochastic State-Continuous Approximation of Markovian Petri Net Systems

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Abstract— **Fluidification constitutes a relaxation technique to study discrete event systems through a continuous approximated model, thus avoiding the state explosion problem. In this paper, the approximation by deterministic Timed Continuous Petri nets under infinite server semantics is studied. The main contribution of this work is the addition of gaussian noise in order to obtain a better (but stochastic) approximation when synchronizations are important.**

I. INTRODUCTION

The computational complexity of analysis and synthesis problems for systems modeled as discrete event makes very important searching for relaxations where computational improvements are significant and, at the same time, the induced errors are small enough to be useful in engineering.

Among the possible relaxations, *fluidification* or *continuization* (i.e. getting state-continuous approximations) is one of the most promising, particularly when the initial state contains many servers and clients. If Petri nets are used, this means that the initial marking can be assumed as "big enough". Two reasons for the above statements: a big marking usually means impossible enumerative computations due to the underlying state explosion problem, and - hopefully - errors tend to become smaller, because the rounding effects are relatively less significant. Continuization relationships were introduced at the net [1] and fundamental equation [2] level. Revisiting the fluidification of timed discrete models, in [3] we propose to use the markovian interpretation of net models as the reference. This leads to the so called infinite-servers semantics, when the number of clients and servers can be considered as "important" and of similar order of magnitude. In particular, if servers (grouped in stations, modeled as transitions in the PN field) are continuously enabled, for long sequences of continuous services the central limit theorem can be applied, and a normal pdf can be assumed with a variance tending to zero, thus leading to a deterministic approximation.

Stochastic Petri nets, under exponential services assumption, straightforwardly lead to markovian net models [4] (in the sequel Markovian Petri nets). In this paper the deterministic approximation of these is enriched by adding some noise to the firing of transitions. If the today classical deterministic state-continuous approximations are usually denoted as TCPN (Timed Continuous Petri Nets) [5], [6], the stochastic approximation will be denoted here as MCPN (Markovian Continuous Petri Nets).

After dealing with basic concepts and notations (sect. II), the deterministic approximation is considered from a new perspective (sect. III). For the quality of the approximation, it is made explicit that servers should be "permanently" active, while the evolution should be "almost" in one region. Later (sect. IV) a new extension, that constitute the central point in this contribution, is considered by adding noise (of null average value) to the TCPN model. The motivation is to improve the approximation of the underlying stochastic process. While the evolution is essentially in a single region, the addition of noise is not relevant (what clearly can be understood from the central limit theorem for extremely long queues). Thus the improvement in the approximation appears when the borders among regions are traversed in the random trajectory. The advantages are on: (1) the approximation of the transient behavior, when such switching among regions appears, and (2) the computation of steady-states, if they are "close" to the border of one region, and evolutions in others are "frequent".

II. BASIC CONCEPTS AND NOTATION

We assume that the reader is familiar with Petri nets (PNs) (for notation we use the standard one, see for instance [7]).

The structure $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post}\rangle$ of *continuous Petri nets* (CPN) is the same as the structure of discrete PNs. That is, P is a finite set of places, T is a finite set of transitions with $P \cap T = \emptyset$, $\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ sized, natural valued, *pre- and post- incidence matrices*. We assume that $\mathcal{N}$ is connected and that every place has a successor, i.e. $|p^\bullet| \geq 1$. The usual PN system, $\langle \mathcal{N}, \mathbf{M}_0\rangle$ with $\mathbf{M}_0 \in \mathbb{N}^{|P|}$, will be said to be *discrete* so as to distinguish it from a *continuous* PN system $\langle \mathcal{N}, \mathbf{m}_0\rangle$, in which $\mathbf{m}_0 \in \mathbb{R}^{|P|}_{\geq 0}$. In the following, the marking of a CPN will be denoted in lower case $\mathbf{m}$, while the marking of the corresponding *discrete* one will be denoted in capital letter $\mathbf{M}$. The main difference between both formalisms is in the evolution rule, since in *continuous* PNs firing is not restricted to be done in integer amounts ([5], [6]). As a consequence the marking is not forced to be integer. More precisely, a transition t is *enabled* at $\mathbf{m}$ iff for every $p \in {}^\bullet t$, $\mathbf{m}(p) > 0$, and its *enabling degree* is $enab(t, \mathbf{m}) = \min_{p \in {}^\bullet t}\{\mathbf{m}(p)/\mathbf{Pre}(p, t)\}$. The firing of t in a certain amount $\alpha \leq enab(t, \mathbf{m})$ leads to a new marking $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token-flow matrix. As in discrete systems, right and left integer annullers of the token flow matrix are called *T-* and *P-flows*, respectively. When they are non-negative, they are called *T-*

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and *P-semiflows*. If there exists $\mathbf{y} > \mathbf{0}$ s.t. $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, the net is said to be *conservative*, and if there exists $\mathbf{x} > \mathbf{0}$ s.t. $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$ the net is said to be *consistent*. Here, we always consider net systems whose initial marking marks all *P-semiflows*.

A markovian stochastic Petri net system *MPN* is a *discrete* system in which the transitions fire at independent exponentially distributed random time delays (see [4]). Then, the firing time of each transition is characterized by its firing rate. In this way, a *MPN* is a tuple $\langle \mathcal{N}, \mathbf{M}_0, \lambda \rangle$, where $\lambda \in \mathbb{R}^{|T|}_{>0}$ represents the transitions rate. Transitions (like station in queueing networks) are the meeting points of clients and servers. In this paper, we will assume infinite-server semantics for all transitions. Then, the time to fire a transition t_i, at a given marking $\mathbf{M}$, is an exponentially distributed r.v. with parameter $\lambda_i \cdot Enab(t_i, \mathbf{M})$, where the enabling degree is forced to be an integer value, i.e. $Enab(t_i, \mathbf{M})$ is the value $\min_{p \in {}^\bullet t_i} \{\mathbf{M}(p)/\mathbf{Pre}(p, t_i)\}$ rounded to the nearest lower integer. $Enab(t_i, \mathbf{M})$ also represents the number of active servers of t_i at marking $\mathbf{M}$. We suppose that a unique steady-state behavior exists. Even more, we restrict to bounded in average and reversible (therefore ergodic) PN systems.

A *Timed Continuous Petri Net (TCPN)* is a *continuous* PN together with a vector $\lambda \in \mathbb{R}^{|T|}_{>0}$. Different semantics have been defined for *continuous* timed transitions, the two most important being *infinite server* or *variable speed*, and *finite server* or *constant speed* (see [5], [6]). Here infinite server semantics will be considered. Like in purely markovian *discrete* net models, under *infinite server semantics*, the flow through a timed transition t_i is the product of the speed, λ_i, and $enab(t_i, \mathbf{m})$, the instantaneous enabling of the transition, i.e., $\mathbf{f}_i(\mathbf{m}) = \lambda_i \cdot enab(t_i, \mathbf{m}) = \lambda_i \cdot \min_{p \in {}^\bullet t_i} \{\mathbf{m}_p / \mathbf{Pre}(p, t_i)\}$. Observe that $Enab(t_i, \mathbf{M}) \in \mathbb{N}$ while $enab(t_i, \mathbf{m}) \in \mathbb{R}_{\geq 0}$. For the flow to be well defined, every transition must have at least one input place, hence in the following we will assume $\forall t \in T, |{}^\bullet t| \geq 1$. The "min" in the definition leads to the concept of *configurations* (see [8]): a configuration assigns to each transition one place that for some markings will control its firing rate. An upper bound for the number of configurations is $\prod_{t \in T} |{}^\bullet t|$. The reachability space can be divided into *marking regions* according to the *configurations*. These *regions* are polyhedrons, and are disjoint, except on the borders.

The flow through the transitions can be written in a vectorial form as $\mathbf{f}(\mathbf{m}) = \mathbf{\Lambda \Pi}(\mathbf{m})\mathbf{m}$, where $\mathbf{\Lambda}$ is a diagonal matrix whose elements are those of λ, and $\mathbf{\Pi}(\mathbf{m})$ is the configuration operator matrix at $\mathbf{m}$, which is defined s.t. the i-th entry of the vector $\mathbf{\Pi}(\mathbf{m})\mathbf{m}$ is equal to the enabling degree of transition t_i (see [8]). A similar representation can also be obtained for the enabling degree of the *discrete* PN, i.e. $Enab(\mathbf{M}) = \lfloor \mathbf{\Pi}(\mathbf{M})\mathbf{M} \rfloor \simeq \mathbf{\Pi}(\mathbf{M})\mathbf{M}$ (the equality is obtained for ordinary PN's, but for weighted arcs, there exists an error for rounding to the nearest lower integer).

The dynamical behavior of a $TCPN$ system is described by its state equation:

$$\dot{\mathbf{m}} = \mathbf{C \Lambda \Pi}(\mathbf{m})\mathbf{m} \tag{1}$$

Inside a given *region*, the state equation is linear because inside this $\mathbf{\Pi}(\mathbf{m})$ is constant.

III. DETERMINISTIC APPROXIMATION

In this section it is shown that the expected value of the marking of a MPN system can be approximated by the marking of the corresponding $TCPN$, always under some particular assumptions. For that, we first approximate the expected value of the making of the MPN via its fundamental equation (i.e. not through the computation of the underlying Markov chain). After that, we show that this expected value can also be approximated by the marking of the $TCPN$ system.

First, consider a MPN system with structure $\mathcal{N}$, timing rates λ, and initial marking M_0. Denote the initial time as τ_0 and consider a particular transition t_i. By definition, at any marking the time to fire each active server of t_i is characterized by a random variable having an exponential p.d.f. with parameter λ_i. Now, consider a fixed time interval $\Delta\tau$. If a server is always active during $\Delta\tau$, then the number of its firings (the number of jobs done) during $\Delta\tau$ is characterized by a r.v. having a Poisson p.d.f. with parameter $\lambda_i \cdot \Delta\tau$ (see [9]). Furthermore, since we are considering infinite server semantics, the number of firings of t_i during $\Delta\tau$ is the sum of the number of firings of each of its servers during this time interval. If $\Delta\tau$ is small enough then the number of active servers of t_i during this time interval remains almost constant. Therefore, if at least one of them is active at $\mathbf{M}_0$ then the number of firings of t_i, during the time interval $(\tau_0, \tau_0 + \Delta\tau)$, can be approximated by a r.v. $\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}_i(\tau_0))$ having a Poisson p.d.f. with parameter $\Delta\mathbf{F}_i(\tau_0) = \Delta\tau \cdot \lambda_i \cdot Enab(t_i, \mathbf{M}_0)$, where $Enab(t_i, \mathbf{M}_0)$ is the number of active servers of t_i at $\mathbf{M}_0$ (the sum of independent Poisson distributed r.v.'s is also a Poisson distributed r.v., whose parameter is the sum of the parameters of the summands).

Notice that we are assuming that $Enab(t_i, \mathbf{M}_0) \geq 1$ and that $\Delta\tau$ is small enough s.t. the probability that $Enab(t_i, \mathbf{M}_0)$ remains constant during $\Delta\tau$ is almost 1, otherwise the approximation of the number of firings of t_i by a Poisson distributed r.v. is not valid.

Now, considering the firing count vector $\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(\tau_0))$, whose elements are the corresponding r.v.'s $\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}_i(\tau_0))$ of each transition, the marking at time $\tau_0 + \Delta\tau$ can be approximated using the fundamental equation, i.e.

$$\mathbf{M}(\tau_0 + \Delta\tau) \simeq \boldsymbol{\mu}(\tau_0 + \Delta\tau) = \mathbf{M}_0 + \mathbf{C}\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(\tau_0)) \tag{2}$$

where $\boldsymbol{\mu}(\tau_0 + \Delta\tau)$ is the approximation of the marking of the MPN at time $\tau_0 + \Delta\tau$.

In the sequel, we will use k to denote $\tau_0 + k\Delta\tau$. In the same way, we will denote by $\boldsymbol{\mu}(k)$ and $\Delta\mathbf{F}(k)$ the approximation of the marking of the MPN at time $\tau_0 + k\Delta\tau$ and the vector function $\Delta\mathbf{F}(\tau_0 + k\Delta\tau) = \Delta\tau \cdot \mathbf{\Lambda} \cdot Enab(\boldsymbol{\mu}(\tau_0 + k\Delta\tau))$, respectively, for any time step k.

Now, $\boldsymbol{\mu}(1)$ is a r.v., whose expected value can be computed as

$$E\{\boldsymbol{\mu}(1)\} = E\{\mathbf{M}_0\} + \mathbf{C} \cdot E\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(0))\}$$

In this case, since $\mathbf{M}_0$ is a known value, then $E\{\mathbf{M}_0\} = \mathbf{M}_0$. Furthermore, $\Delta\mathbf{F}(0)$ is also a deterministic value, then $E\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(0))\} = \Delta\mathbf{F}(0) = \Delta\tau \cdot \boldsymbol{\Lambda} \cdot Enab(\mathbf{M}_0)$. Now, since $Enab(\mathbf{M}_0) \simeq \boldsymbol{\Pi}(\mathbf{M}_0)\mathbf{M}_0$ (remember that it is assumed to be ≥ 1, moreover, the equality holds for ordinary nets), then $E\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(0))\} \simeq \boldsymbol{\Lambda\Pi}(\mathbf{M}_0)\mathbf{M}_0\Delta\tau$. Therefore

$$E\{\boldsymbol{\mu}(1)\} \simeq [\mathbf{I} + \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{M}_0)\Delta\tau]\mathbf{M}_0 \tag{3}$$

Now, let us try to approximate the expected value for the marking at time step 2. By the memoryless property of the MPN system, (2) can also be used for the next time step, then

$$\mathbf{M}(\tau_0 + 2\Delta\tau) \simeq \boldsymbol{\mu}(2) = \boldsymbol{\mu}(1) + \mathbf{C}\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1)) \tag{4}$$

and also

$$\begin{array}{c} E\{\mathbf{M}(\tau_0 + 2\Delta\tau)\} \simeq E\{\boldsymbol{\mu}(2)\} \\ = E\{\boldsymbol{\mu}(1)\} + \mathbf{C} \cdot E\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1))\} \end{array} \tag{5}$$

Now, notice that $E\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1))\}$ cannot be computed as before, since $\boldsymbol{\mu}(1)$ is a r.v. with unknown p.d.f., and so it is $\Delta\mathbf{F}(1)$. However, obtaining the conditional expected value

$$\begin{array}{c} E\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1))\} = \\ \sum_{\boldsymbol{\mu}} E\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1)) | \boldsymbol{\mu}(1) = \boldsymbol{\mu}\} \cdot Prob(\boldsymbol{\mu}(1) = \boldsymbol{\mu}) \\ \simeq \sum_{\boldsymbol{\mu}} \boldsymbol{\Lambda\Pi}(\boldsymbol{\mu})\boldsymbol{\mu}\Delta\tau \cdot Prob(\boldsymbol{\mu}(1) = \boldsymbol{\mu}) \\ = E\{\boldsymbol{\Lambda\Pi}(\boldsymbol{\mu}(1))\boldsymbol{\mu}(1)\Delta\tau\} \end{array} \tag{6}$$

As already considered for $\mathbf{M}_0$, we have to assume that all the transitions are enabled at $\boldsymbol{\mu}$ in order to approximate the number of firings during $\Delta\tau$ at $\boldsymbol{\mu}$ by a Poisson distributed r.v.. This assumption must be true at least for the most probable values of $\boldsymbol{\mu}(1)$. This is generalized by the following condition:

Condition 1. Given a time step k, the probability that the transitions are all enabled at marking $\boldsymbol{\mu}(k)$ is near one.

Moreover, an additional assumption that the net system probabilistically remains in the region defined by $\mathbf{M}_0$ is technically stated as:

Condition 2. Given a time step k, the probability that the marking $\boldsymbol{\mu}(k)$ is outside the marking region of $\mathbf{M}_0$ is near zero.

If condition 2 is fulfilled for $k = 1$, then $\boldsymbol{\Pi}(\boldsymbol{\mu}(1)) = \boldsymbol{\Pi}(\mathbf{M}_0)$ and so

$$\begin{array}{c} E\{\boldsymbol{\Lambda\Pi}(\boldsymbol{\mu}(1))\boldsymbol{\mu}(1)\Delta\tau\} = E\{\boldsymbol{\Lambda\Pi}(\mathbf{M}_0)\boldsymbol{\mu}(1)\Delta\tau\} \\ = \boldsymbol{\Lambda\Pi}(\mathbf{M}_0)\Delta\tau E\{\boldsymbol{\mu}(1)\} \end{array}$$

So, substituting into (5), we obtain that

$$\begin{array}{c} E\{\mathbf{M}(\tau_0 + 2\Delta\tau)\} \simeq E\{\boldsymbol{\mu}(2)\} \simeq [\mathbf{I} + \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{M}_0)\Delta\tau] \cdot \\ \cdot E\{\boldsymbol{\mu}(1)\} \simeq [\mathbf{I} + \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{M}_0)\Delta\tau]^2\mathbf{M}_0 \end{array}$$

Finally, following and inductive reasoning, we obtain that

$$\begin{array}{c} E\{\mathbf{M}(\tau_0 + n\Delta\tau)\} \simeq E\{\boldsymbol{\mu}(n)\} \\ \simeq [\mathbf{I} + \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{M}_0)\Delta\tau]^n\mathbf{M}_0 \end{array} \tag{7}$$

where it is assumed that conditions 1 and 2 are always fulfilled during the time interval $(\tau_0, \tau_0 + n\Delta\tau)$.

Now, consider the $TCPN$ system given by $\langle N, \lambda, \mathbf{m}_0 \rangle$, and its state equation (1). The corresponding discrete-time model (see, for example, [10]), taking $\Delta\tau$ as the sampling period, is given by

$$\begin{array}{c} \mathbf{m}_{k+1} = \mathbf{A}_D\mathbf{m}_k \\ where\ \mathbf{A}_D = \sum_{r=0}^{\infty} \frac{(\mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{m}_k)\Delta\tau)^r}{r!} \end{array}$$

For a small enough $\Delta\tau$, $\mathbf{A}_D \simeq \mathbf{I} + \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{m}_k)\Delta\tau$. Therefore, $\mathbf{m}_{k+1}$ is approximated by

$$\mathbf{m}_{k+1} \simeq \mathbf{m}_k + \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau \tag{8}$$

Comparing this equation (for $k = 0$) with (3), it can be concluded that, if $\mathbf{M}_0 = \mathbf{m}_0$ and the condition 1 is fulfilled, then $E\{\mathbf{M}(1)\} \simeq \mathbf{m}_1$. Moreover, considering the condition 2, $E\{\boldsymbol{\mu}(1)\} \simeq E\{\mathbf{M}(1)\}$ belongs to the same region of $\mathbf{M}_0$, so, $\mathbf{m}_1$ will be quite probably in the same one. Therefore, the marking at time 2 of the $TCPN$ system can be estimated by using the same difference equation (i.e. (8) with $k = 1$ and $\boldsymbol{\Pi}(\mathbf{m}_1) = \boldsymbol{\Pi}(\mathbf{m}_0)$). Following an inductive reasoning over the time interval $(\tau_0, \tau_0 + n\Delta\tau)$, we obtain:

$$\mathbf{m}_n \simeq [\mathbf{I} + \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{m}_0)\Delta\tau]^n\mathbf{m}_0$$

Comparing this equation with (7), it can be concluded that the marking trajectory of the $TCPN$ system approximates the expected value of the marking of the corresponding MPN if the conditions 1 and 2 are fulfilled during $(\tau_0, \tau_0 + n\Delta\tau)$.

For instance, consider the MPN of figure 1(a) with firing rates $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The $TCPN$ system has been simulated, by using MATLAB, and the steady-state results have been compared with those obtained for the MPN using TimeNET [11]. Table I resumes the results thus obtained. The first column represents the initial marking, given by $\mathbf{M}_0 = \mathbf{m}_0 = q \cdot [1, 1, 1]^T$. Second and third columns are the expected value of the MPN at the steady state, and the final value of the $TCPN$, respectively, for place p_2. The fourth column is the marking error and the final column is the probability that all the transitions are enabled at the steady state (for the *discrete* system).

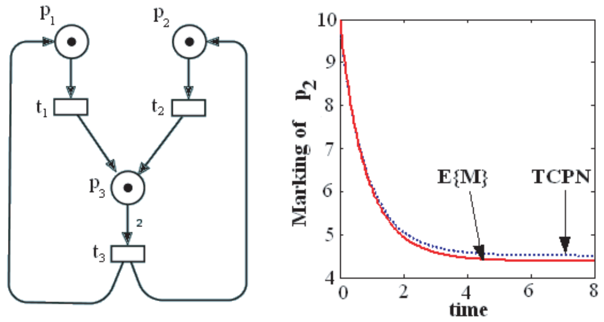
Since there is no synchronization in this net, condition 2 is always fulfilled. A value of 1 in the fifth column means that the first condition is fulfilled. As expected, the error is lower when this value approximates 1, which occurs for large values of q. The figure 1(b) shows the evolution of both the marking of the $TPCN$ and the expected value of the marking of the MPN, at place p_2, for the initial marking $\mathbf{M}_0 = 6 \cdot [1, 1, 1]^T$. As it can be seen, the transient behavior of the MPN is also well approximated by the $TCPN$.

Now, consider the MPN system of figure 2, and its corresponding $TCPN$ one, with initial marking $\mathbf{M}_0 = \mathbf{m}_0 = [5, 5, 55, 5, 6, 4]^T$ and timing rates $\lambda_1 = \lambda_2 = \lambda_3 = 1$ for the first three transitions. Both MPN and $TCPN$ are studied for different values of λ_4. The results are shown in Table II. The values of the second and third columns correspond to the steady state marking of place p_3. The value of column $P.C1$ is the probability that all the transitions are enabled at the steady state, i.e. that the condition 1 is fulfilled, while the value at the last column ($P.C2$) is the probability

TABLE I

MARKING APPROXIMATION OF p_2 FOR THE MPN IN FIG. 1(A)

q	MPN	TCPN	error	P. C1
3	2.126	2.25	5.81%	0.669
4	2.878	3.00	4.25%	0.844
6	4.370	4.50	2.98%	0.987
9	6.639	6.75	1.67%	0.998
12	8.911	9.00	0.99%	0.997
15	11.911	12.00	0.75%	0.999



(a) (b)

Fig. 1. (a) A join-free, but not ordinary, Petri Net System. (b) In solid line, the evolution of the expected value of the marking at place p_2. In dashed line, the corresponding marking of the TCPN.

that the marking is inside the marking region of $\mathbf{M}_0$, i.e. that condition 2 is fulfilled.

It can be observed that, the lower the probability that condition 2 is fulfilled, the bigger the error in the marking, even if the probability that condition 1 is fulfilled increases. This example shows the importance of condition 2.

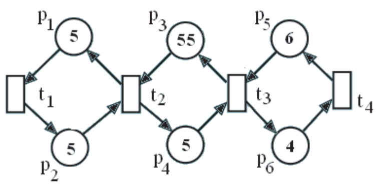


Fig. 2. Marked graph in which condition 2 is not fulfilled

IV. STOCHASTIC APPROXIMATION BY ADDING NOISE TO THE TCPN SYSTEM

In this section, an addition to the $TCPN$ model is introduced in order to improve the approximation, even if

TABLE II

MARKING APPROXIMATION OF p_3 FOR THE MPN OF FIG. 2

λ_4	MPN	TCPN	error	P. C1	P. C2
2	54.62	55	0.7%	0.812	0.774
1.5	53.87	55	2.1%	0.889	0.576
1.2	51.16	55	7.5%	0.930	0.306
1.1	46.65	55	17.9%	0.941	0.207
1.05	40.72	55	35.0%	0.956	0.040

condition 2 is not fulfilled. For this, we first estimate the moments of the MPN system, next, we propose a modified $TCPN$ system, by adding noise, whose moments coincide with those of the MPN one.

Then, let us focus on the approximation of the moments of the marking. As shown in the previous section, the marking of the MPN at time $\tau_0+\Delta\tau$ is approximated by (2). Since $\mathbf{M}_0$ is a deterministic value and the entries of $\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(0))$ are independent Poisson r.v.'s, then the mean and covariance matrix of $\boldsymbol{\mu}(1) \simeq \mathbf{M}(\tau_0+\Delta\tau)$ can be easily computed. For the next time step, we obtained (4). According to this equation, in order to approximate the mean and covariance matrix of $\boldsymbol{\mu}(2) \simeq \mathbf{M}(\tau_0+2\Delta\tau)$ it is necessary to estimate the mean and covariance of $\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1))$.

As already observed, the firing count $\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1))$ is Poisson distributed with parameter $\Delta\mathbf{F}(1)$, but now this parameter is also a r.v.. However, the mean and variance of the firing count can be expressed in terms of the moments of its parameter, as shown next for a particular transition t_i (first equation is equivalent to (6), both are recalled from [9])

$$E\{\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}_i(1))\} = E\{\Delta\mathbf{F}_i(1)\} \tag{9}$$

$$var\{\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}_i(1))\} = var\{\Delta\mathbf{F}_i(1)\} + E\{\Delta\mathbf{F}_i(1)\} \tag{10}$$

Furthermore, using the total probability theorem and some properties of the conditional expected value, it can be demonstrated, for any other transition t_j or place p_j, that

$$\begin{aligned} &cov\{\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}_i(1)), \Delta\boldsymbol{\sigma}_j(\Delta\mathbf{F}_j(1))\} \\ &\quad = cov\{\Delta\mathbf{F}_i(1), \Delta\mathbf{F}_j(1)\} \end{aligned} \tag{11}$$

$$cov\{\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}_i(1)), \boldsymbol{\mu}_j(1)\} = cov\{\Delta\mathbf{F}_i(1), \boldsymbol{\mu}_j(1)\} \tag{12}$$

Then, denoting the covariance matrices of $\Delta\mathbf{F}(1)$ and $\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1))$ with $\boldsymbol{\Sigma}_{\Delta\mathbf{F}(1)}$ and $\boldsymbol{\Sigma}_{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1))}$, respectively, it is easy to see, by using (10) and (11), that

$$\boldsymbol{\Sigma}_{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(1))} = \boldsymbol{\Sigma}_{\Delta\mathbf{F}(1)} + diag\left[E\{\Delta\mathbf{F}(1)\}\right] \tag{13}$$

where $diag[\mathbf{a}]$ is a diagonal matrix whose diagonal elements are the corresponding entries of $\mathbf{a}$.

Therefore, with all these equations, we can express the moments of $\boldsymbol{\mu}(2)$ as functions of the moments of $\boldsymbol{\mu}(1)$ and $\Delta\mathbf{F}(1)$. In general, these equations allow us to express the moments of $\boldsymbol{\mu}(k+1)$ as functions of the moments of $\boldsymbol{\mu}(k)$ and $\Delta\mathbf{F}(k)$, for any time step k, in the following way

$$E\{\boldsymbol{\mu}(k+1)\} = \begin{bmatrix} \mathbf{I} & \mathbf{C} \end{bmatrix} \begin{bmatrix} E\{\boldsymbol{\mu}(k)\} \\ E\{\Delta\mathbf{F}(k)\} \end{bmatrix}$$

$$\begin{aligned} &\boldsymbol{\Sigma}_{\boldsymbol{\mu}(k+1)} = \\ &\begin{bmatrix} \mathbf{I} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{\mu}(k)} & \boldsymbol{\Sigma}_{\boldsymbol{\mu}(k),\Delta\boldsymbol{\sigma}(\mathbf{k})} \\ \boldsymbol{\Sigma}_{\Delta\boldsymbol{\sigma}(\mathbf{k}),\boldsymbol{\mu}(k),\Delta\tau} & \boldsymbol{\Sigma}_{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k))} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{C}^T \end{bmatrix} \end{aligned}$$

where $\boldsymbol{\Sigma}_{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k))}$ is given by (13), and the cross covariance matrices $\boldsymbol{\Sigma}_{\boldsymbol{\mu}(k),\Delta\boldsymbol{\sigma}(k)}$ and $\boldsymbol{\Sigma}_{\Delta\boldsymbol{\sigma}(k),\boldsymbol{\mu}(k)}$ can be computed by elements using (12), i.e. the element at the i-th row and j-th column of $\boldsymbol{\Sigma}_{\boldsymbol{\mu}(k),\Delta\boldsymbol{\sigma}(k)}$ is given by $cov(\boldsymbol{\mu}_i(k), \Delta\boldsymbol{\sigma}_j(\Delta\mathbf{F}_j(k))) = cov(\boldsymbol{\mu}_i(k), \Delta\mathbf{F}_j(k))$.

Next, we propose a modification to the $TCPN$ deterministic model. Consider again the discrete-time version of the

$TCPN$ system, given by (8). Let us define a noise column vector $\mathbf{v}_k$, of length $|T|$, whose elements are independent normally distributed r.v.'s with mean and covariance matrix:

$$\begin{array}{c} E\{\mathbf{v}_k\} = \mathbf{0} \\ \mathbf{\Sigma}_{\mathbf{v}_k} = diag[\mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau] = diag[\mathbf{f}_k\Delta\tau] \end{array} \tag{14}$$

Then, the discrete-time version of the $TCPN$ system, with the noise $\mathbf{v}_k$ being added to the firing count, is given by the following difference equation:

$$\begin{array}{c} \mathbf{m}_{k+1} \simeq \mathbf{m}_k + \mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau + \mathbf{C}\mathbf{v}_k \\ = \mathbf{m}_k + \mathbf{C}\mathbf{w}_k \end{array} \tag{15}$$

where $\mathbf{w}_k = \mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau + \mathbf{v}_k = \mathbf{f}_k\Delta\tau + \mathbf{v}_k$.

In the sequel, this modified $TCPN$ will be called *markovian* continuous Petri net ($MCPN$) so as to distinguish it from the original *deterministic* one. Now, in order to show that this system approximates the MPN one, let us follow an inductive reasoning. First, we assume that the initial marking is known, so we set $\mathbf{m}_0 = \mathbf{M}_0$. Now, suppose that at some time $\tau_0 + k\Delta\tau$, the marking of the MPN system $\mathbf{M}(\tau_0 + k\Delta\tau) \simeq \boldsymbol{\mu}(k)$ is well approximated by $\mathbf{m}_k$, so the mean fulfills $E\{\mathbf{m}_k\} \simeq E\{\boldsymbol{\mu}(k)\}$ and the covariance matrix fulfills $\mathbf{\Sigma}_{\mathbf{m}_k} \simeq \mathbf{\Sigma}_{\boldsymbol{\mu}(k)}$. In this way, the r.v. $\Delta\mathbf{F}(k)$ is well approximated by the r.v. $\mathbf{f}_k\Delta\tau$ (i.e. the r.v.'s obtained by applying a similar function are similar). Furthermore, with $\mathbf{f}_k$ being a r.v., it can be demonstrated that $\mathbf{\Sigma}_{\mathbf{v}_k} = diag[E\{\mathbf{f}_k\Delta\tau\}]$. Then, using (9) and (13), it is easy to see that

$$\begin{array}{c} E\{\mathbf{w}_k\} = E\{\mathbf{f}_k\Delta\tau\} \simeq E\{\Delta\mathbf{F}(k)\} = E\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k))\} \\ \mathbf{\Sigma}_{\mathbf{w}_k} = \mathbf{\Sigma}_{\mathbf{f}_k\Delta\tau} + diag[E\{\mathbf{f}_k\Delta\tau\}] \\ \simeq \mathbf{\Sigma}_{\Delta\mathbf{F}(k)} + diag[E\{\Delta\mathbf{F}(k)\}] = \mathbf{\Sigma}_{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k))} \end{array}$$

In the same way, for any transition t_i and place p_j it is easy to demonstrate, using (12), that

$$\begin{array}{c} cov(\mathbf{w}_{ki}, \mathbf{m}_{kj}) = cov(\mathbf{f}_{ki}\Delta\tau, \mathbf{m}_{kj}) \\ \simeq cov(\Delta\mathbf{F}_i(k), \boldsymbol{\mu}_j(k)) = cov(\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}_i(k)), \boldsymbol{\mu}_j(k)) \end{array}$$

Therefore, the mean and covariance matrix of $\mathbf{M}(\tau_0+(k+1)\Delta\tau) \simeq \boldsymbol{\mu}(k+1)$ are similar to those of $\mathbf{m}_{k+1}$. Now, in order to show that they also have a similar p.d.f., suppose that the p.d.f. of the marking of the MPN for the previous s steps are well approximated, and let $n = k - s$. Then, iterating (3) we obtain

$$\boldsymbol{\mu}(k+1) \simeq \boldsymbol{\mu}(n) + \mathbf{C} \cdot \sum_{r=0}^{s} \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(n+r))$$

By the total probability theorem

$$\begin{array}{c} \sum_{r=0}^{s} \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(n+r)) \\ \simeq \sum_{\boldsymbol{\mu}} \sum_{r=0}^{s} \{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(n+r)) | \boldsymbol{\mu}(n+r) = \boldsymbol{\mu}\} \\ \cdot Prob(\boldsymbol{\mu}(n+r) = \boldsymbol{\mu}) \end{array}$$

where the sum $\sum_{\boldsymbol{\mu}}$ consider all the possible values of $\boldsymbol{\mu}$ during the time interval (n, k). Now, for each fixed $\boldsymbol{\mu}$, the entries of $\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(n+r)) | \boldsymbol{\mu}(n+r) = \boldsymbol{\mu}\}$ are independent Poisson distributed r.v.'s (independent for different time steps). So, by the Central Limit Theorem (see, for example, [12]), for a large enough s, the entries of the sum $\sum_{r=0}^{s} \{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(n+r)) | \boldsymbol{\mu}(n+r) = \boldsymbol{\mu}\} \cdot Prob(\boldsymbol{\mu}(n+r) = \boldsymbol{\mu})$ can be considered as normal distributed r.v.'s. Actually, the distribution of each entry of this sum converges very fast to a normal p.d.f. (in few time steps), because $Prob(\boldsymbol{\mu}(n+r) = \boldsymbol{\mu})$ is almost constant for consecutive time steps and the r.v.'s $\{\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(n+r)) | \boldsymbol{\mu}(n+r) = \boldsymbol{\mu}\}$ are identically distributed, which implies that the variances of the summands are similar (so they fulfill the Lindeberg condition for the proof of the Central Limit Theorem [12]).

Similarly, for the $MCPN$, we obtain

$$\begin{array}{c} \mathbf{m}_{k+1} \simeq \mathbf{m}_n + \mathbf{C} \cdot \sum_{r=0}^{s} \mathbf{w}_{n+r} \\ \sum_{r=0}^{s} \mathbf{w}_{n+r} \simeq \sum_{\mathbf{m}} \sum_{r=0}^{s} \{\mathbf{w}_{n+r} | \mathbf{m}_{n+r} = \mathbf{m}\} \\ \cdot Prob(\mathbf{m}_{n+r} = \mathbf{m}) \end{array}$$

By definition of $\mathbf{v}_k$ (14) and $\mathbf{w}_k$ (15), the entries of $\{\mathbf{w}_{n+r} | \mathbf{m}_{n+r} = \mathbf{m}\}$ are independent normally distributed r.v.'s (independent for different time steps). In the same way, the entries of the vector $\sum_{r=0}^{s} \{\mathbf{w}_{n+r} | \mathbf{m}_{n+r} = \mathbf{m}\} \cdot Prob(\mathbf{m}_{n+r} = \mathbf{m})$ are also normally distributed r.v.'s. Therefore, the distributions of $\sum_{r=0}^{s} \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(n+r))$ and $\sum_{r=0}^{s} \mathbf{w}_{n+r}$ are similar (their summands have similar distributions). Moreover, if condition 1 is fulfilled, they approximate the distribution of the sum of the corresponding firing counts of the MPN. Then, the p.d.f. of $\mathbf{M}(\tau_0+(k+1)\Delta\tau) \simeq \boldsymbol{\mu}(k+1)$ and $\mathbf{m}_{k+1}$ are similar. Finally, let us point out that the hypothesis considered about the approximation for the previous s steps is nearly true for the first time steps, this is because the initial marking is a known deterministic value and s is not required to be large, which implies that the distributions for the first s markings are close to their means (variances are very low).

Therefore, it can be concluded that, considering a time interval $(\tau_0, \tau_0 + k\Delta\tau)$, the moments of the marking of the $MCPN$ system (15) approximate the moments of the marking of the corresponding MPN, if the initial conditions of both coincide, and condition 1 is fulfilled during $(\tau_0, \tau_0 + k\Delta\tau)$.

For instance, consider again the MPN of figure 2. Simulations of the corresponding $MCPN$ were made, by using MATLAB, for the same values of initial marking and timing rates. For each value of λ_4 a simulation was obtained for a very large time, so, the mean value of these correspond to the expected value at the steady states. These means are shown in Table III (third column). Comparing the errors thus obtained (fourth column), with those of the $TCPN$ (showed in II), it can be seen that the approximation by $MCPN$ systems is much better.

Furthermore, for timing rates $\lambda_1 = \lambda_2 = \lambda_3 = 1, \lambda_4 = 2$ and initial marking $\mathbf{M}_0 = [1, 9, 1, 59, 1, 9]^T$, the expected value of the marking of the MPN system is computed by using TimeNET. Also, the MPN system and the corresponding $MCPN$ are simulated 30 times with MATLAB. Figure 3 shows the evolution of the marking of p_3. The smooth curves correspond to the expected value of the MPN (denoted as $E\{M\}$) and to the original $TCPN$ system (denoted as $TCPN$). The other curves correspond to the average means

TABLE III

MARKING APPROXIMATION OF p_3 FOR THE MPN OF FIG. 2 VIA ITS CORRESPONDING $MCPN$

λ_4	MPN	MCPN	error
2	54.62	54.63	0.26%
1.5	53.87	53.88	0.22%
1.2	51.16	51.17	0.49%
1.1	46.65	46.66	1.52%
1.05	40.72	40.73	1.78%

TABLE IV

APPROXIMATION OF p_5 IN STEADY STATE FOR THE MPN OF FIG. 4(A)

$\mathbf{M}_0(p_4)$ $+\mathbf{M}_0(p_5)$	MPN	MCPN	error	P. C1
15	3.71	3.68	0.7%	0.929
20	4.41	4.22	4.2%	0.912
30	4.24	3.96	6.7%	0.854
40	3.65	3.21	11.9%	0.707
50	3.34	2.62	21.5%	0.534

of 30 realizations of both the MPN (denoted by MPN) and the $MCPN$ system (denoted by $MCPN$). Vertical lines represent changes of regions. As it can be seen, the average means of the realizations of both systems MPN and $MCPN$ are similar, and close to the expected value of the marking. On the other hand, the marking of the $TCPN$ system shows a more significant error (of 8.8% at time $\tau = 32$), when the second change of region occurs.

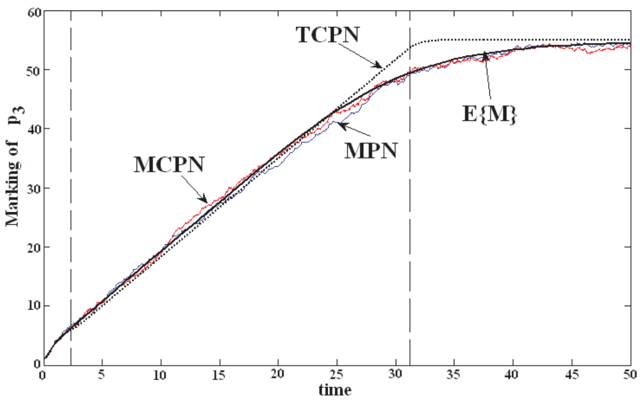
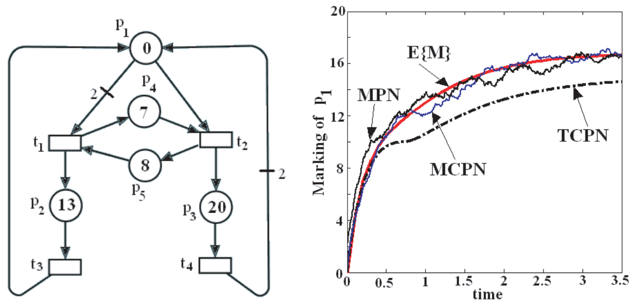


Fig. 3. Approximation of the transient behavior of the MPN of fig. 2 for $\lambda_4 = 2$.

Now, consider the MPN of figure 4(a), with timing rates $\lambda_1 = 3, \lambda_2 = \lambda_3 = \lambda_4 = 1$ and initial marking $\mathbf{M}_0 = [0, 13, 20, 7, 8]^T$. The expected value of the marking at the steady state, for different values of the initial marking at place p_5, is computed by using TimeNET. Also, for each value of p_5 (at the initial marking), a simulation of the corresponding $MCPN$ was obtained for a very large time with MATLAB. The results obtained are shown in Table IV. The first column corresponds to the value of $\mathbf{M}_0(p_4)+\mathbf{M}_0(p_5)$ at the initial marking. The second column shows the expected value of the marking of the MPN for place p_5 at the steady state, while the third one shows the means of the simulations of the corresponding $MCPN$. The fourth column corresponds to the marking error and the fifth column is the probability that condition 1 is fulfilled.

In this net, the relationship between the initial marking and the corresponding throughput, studied in [13], is non monotonic. In that paper it was concluded that its corresponding $TCPN$ (without noise) does not provide a good approximation. However, in Table IV it can be seen that the $MCPN$ does provide a good approximation for the initial conditions in which $\mathbf{M}_0(p_4) + \mathbf{M}_0(p_5)$ ranges from 15 to 30. As it is expected, the larger is the probability that condition 1 is fulfilled, the better the approximation. Furthermore, the transient behavior can also be well approximated. Figure 4(b) shows the evolution of the marking of place p_1 during the first 3.5 seconds, for the initial marking $\mathbf{M}_0 = [0, 13, 20, 7, 8]^T$. The expected value of the MPN, obtained by using TimeNET, is denoted by $E\{\mathbf{M}\}$, while the marking of the original $TCPN$ system is denoted by $TCPN$. The mean trajectory of 100 simulations of the MPN and the $MCPN$, obtained with MATLAB, are also shown in fig. 4(b), where are denoted by MPN and $MCPN$, respectively. Notice that the curve $MCPN$ is always close to the curves $E\{\mathbf{M}\}$ and MPN, furthermore, $MCPN$ provides a better approximation than $TCPN$ (the approximation by the $TCPN$ is not so good because there are several regions).



(a) (b)

Fig. 4. (a) Petri Net System. (b) Approximation of the transient behavior of the marking of place p_1. Region commutations in the $TCPN$ occur at times 0.253, 0.715, 0.718 and 1.137.

V. CONCLUSIONS

Using the fundamental equation, we have shown that a MPN system can be approximated by its corresponding $TCPN$ one, whenever the transitions are enabled and the marking remains inside the same region. Moreover, by adding gaussian noise to the $TCPN$ system, we have obtained a new model $MCPN$ which approximates the MPN considering different regions. This result is very interesting, since it constitutes a bridge between MPN systems and state-continuous ones. Then, the MPN system can be now studied by using the tools developed in Control theory for state-continuous systems.

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